Malaysian Journal of Mathematical Sciences 20(1): 293–324 (2026)
https://doi.org/10.47836/mjms.20.1.15

Malaysian Journal of Mathematical Sciences

Journal homepage: https://mjms.upm.edu.my

A Study on the Fractional-Order Linear Dynamical System with Time-Varying Delays and Fracture

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Received: 5 December 2024
Accepted: 13 August 2025

Abstract

This article is about the study of a novel concept of finite-time stability of a fractional-order dynamical system with time-varying delays and fracture. We have obtained the solution and established stability conditions of the system. Moreover, the controllability and observability conditions in the form of Gramian matrices and matrix conditions are derived based upon the well-known Kalman rank criteria. The results obtained are novel, and to facilitate a better understanding, we have included several illustrative examples.

Keywords: Caputo fractional derivative; time-varying delay; fracture; finite-time stability; controllability and observability.

1 Introduction

Fractional calculus (FC), also known as arbitrary order calculus, exists simultaneously with classical calculus. It has its origin in the 1600s [35]. A few decades ago, researchers discovered that irregularities that could not be explained by classical calculus were easily explained by FC. It is a valuable tool for describing abnormal diffusion processes due to its numerous characteristics, such as memory and hereditary properties. FC has a wide range of applications, including electrochemical processes [26], viscoelastic materials [51], blood flow phenomena, mathematical biology such as fractional Possum model [22], fractional order dengue disease model [30], model of HIV [62] and Chikungunya model [63]. Additionally, a variety of problems can be solved using FC, and it is effectively used in the modeling of many processes [64]. Moreover, the fractional-order models give more information and detailed results, which are useful in demonstrating how challenging and commonplace the order of the derivative is because it has many alternatives [23].

Delays are a common cause of destabilization and performance degradation in practical scenarios such as chemical processes, communications transactions, and neural networks. This has aroused growing interest in recent decades [15]. Delays are found in electrical, pneumatic, and hydraulic networks, chemical processes, and long transmission lines. The presence of pure time delay may result in instability [17]. The stability of dynamical systems with time delays has become an important research topic as they occur in many systems where they cause instability and oscillation [19]. The issue of instability and poor performance of dynamical systems caused by time delays has attracted an immense amount of attention and research. Meanwhile, fractional-order systems (FOS) have gained prominence due to the numerous benefits of fractional derivatives (FDs). According to [46, 55], the model provides researchers with greater flexibility and incorporates the memory of different materials and processes. The finite-time stability (FTS) concept initially emerged in the 1950s. It differs from classical stability in two key aspects. Firstly, it addresses systems that operate within a fixed time interval, which is finite. Secondly, the system variables remain within prescribed bounds in FTS.

For systems operating within a finite time interval, the FTS criterion is the most appropriate one [20]. The key characteristic of a control system is its stability, which indicates the system's ability to get back to the desired state after being interrupted. The most significant quality of control systems is their stability, which ensures that the output is under control. Stability is defined by the roots of the control system's characteristic equation in the complex domain [2]. One of the primary concerns in control theory is the stability analysis of fractional-order time-varying systems (FOTVS), and determining the circumstances under which the system is stable is an extremely difficult task. To check the stability of fractional dynamical systems, several mathematical tools have been devised and applied which include the linear matrix inequality (LMI). Stability analysis conditions are proposed through a direct stability domain characterization [58], for the $n-$dimensional fractional differential equation (FDE) with multiple time delays, the Laplace transform method [18] is utilized and a characteristic equation is introduced, the Lyapunov method [48] and the definition of Mittag-Leffler stability was proposed [73]. Singular systems have drawn considerable attention in recent times. They are widely applied to engineering systems, biological systems, and economic systems [16]. In 2009, Haidar and Boukas [25] and in 2011, Lin et al. [49] considered the singular system along with time-varying delays. The exponential admissibility of both these systems is investigated in [70].

In solving dynamical systems with fractional-order, the Mittag-Leffler function (MLF) is a key factor. It plays an important part in finding the solution of non-integer order differential equations (DEs), similar to the exponential, which naturally originates in the solution of integer order DEs [41]. The asymptotical stability results are obtained using the frequency domain approach in

[6, 7]. Several reports have been published on time delay systems (TDS) for stability, using matrix measure techniques or Lyapunov's second method [10, 27]. The system's stability in the sense of a non-Lyapunov is studied in [44, 69]. In the real scenario, interest often extends beyond just finding the system's stability (as defined by Lyapunov), to including the bounds on the system trajectories. It is valuable to analyze the system's stability, which can be stable but still unstable if it exhibits unacceptable transitory performances. Furthermore, focusing on the dynamical system behavior over finite time intervals is essential. Engineering depends heavily on the boundedness characteristics of system responses, sometimes known as model solutions [45].

Recent advancements in control theory for fractional dynamical systems have addressed stability concerns. Matignon [52, 53] investigates internal and external stabilities for linear fractional differential systems in the state-space form of finite dimensions. A condition based on an argument principle ensures the asymptotic stability of the system with fractional-order. The robust stability results of the systems with fractional-order are presented and discussed in [12, 65]. Chen and Moore [13, 14] proposed an analytical approach for determining the analytical stability bound for ordinary/fractional-order delay differential equations (FO-DDEs) with the aid of the Lambert W function. Additionally, research has focused on the analysis and stabilization of fractional delay systems, specifically of the retarded/neutral types [5] and BIBO stability [28]. Notably, the initial study addressing FTS of fractional time-delay systems (FTDS) is studied in [42].

Fracture has been an ongoing problem in real life. Advances in fracture mechanics have reduced the dangers associated with increasing technological complexity. Since World War II (WWII), we have developed a greater knowledge of material failures and increased our ability to safeguard against them. There is still a lot to be learned about fracture mechanics, and current knowledge is not always used effectively. Advances in computer technology have created entirely new disciplines of fracture mechanics research. The microelectronics industry's problems led to research on interfacial and nanoscale fractures [3]. Understanding fracture mechanics and SIF is important for predicting failure when a crack propagates. Engineers can improve safety and reduce risks by detecting possible failures and influencing factors, especially in areas like civil engineering [34] and aerospace [36].

The optimal solution for control challenges is the primary objective of a control system. The first objective is the ability to transfer any initial state in a finite amount of time to a desired condition using a suitable control input for any system. The second approach involves determining the system's initial stage based on finite-time output under the knowledge of the input. Both of these objectives are linked to the concepts of controllability and observability [9]. Mathematicians and engineers have been attracted to the concepts of controllability and observability (being the control properties) as they are important in the field of control theory and engineering [50]. In 2022, Šajić et al. [59] presented biomedical engineering applications of control system theory. In the same year, Wang et al. [66] examined the properties of controllability and observability of mixed traffic systems. Impulsive fractional LTI systems were considered by Guo [24], where the controllability and observability of such systems is investigated.

Wei [68] in 2012, analyzed the controllability of control systems with fractional-order and control delay. Singh and Pandey [61] presented findings on the time-fractional differential system, where they explored the exact controllability. Controllability, as well as observability and stability, are some essential concepts that play an integral role in both classical and fractional-order mathematical control theory. In recent times, the controllability of control systems with time delay has been addressed in several monographs and studies. This has been inspired by the large range of potential applications in numerous areas of science and engineering, as well as by the diverse and challenging theoretical problems that such systems pose [31]. Controllability of linear systems with different types of delays is considered in various monographs like [33] and survey papers

[39, 40].

Moreover, in the last 20 years, numerous studies have focused on the importance of time delays in industrial systems, including chemical reactors, airplane stabilization, neural networks, microwave oscillation, population dynamic models, long transmission lines, and manual control. Typically, time delay (TD) is an unstable factor. As a result, numerous studies have been conducted to explore the stability and stabilization of TDS [8, 60]. Significant difficulties arise in system design and control when dealing with time-varying delays, fractures, and disturbances. It is essential to comprehend their effects, model them precisely, and create efficient compensation plans to preserve system stability and performance.

In light of this literature survey, our motivation and work objectives are centered on addressing a new problem of fractional-order dynamical systems with time-varying delays and fracture, aiming to explore their theoretical foundations and practical implications. By doing so, we seek to contribute to a deeper understanding of dynamical systems of fractional-order, which will showcase their applicability across various fields, such as engineering, biology, robotics, aerospace, and control systems. This research strives to bridge the gap between theoretical advancements and real-world applications, ultimately fostering innovation and solving complex problems in these domains.

2 Novelty

This work mainly focuses on non-integer-order time-varying delay systems under a fractional-order approach to discuss the solution, its stability, controllability, and observability.

In dynamical systems with time-varying delays, the effects of stress can lead to structural failure and instability if it is not managed properly. So, we have considered a fractional-order dynamical system defined as:

$$\begin{cases} {}^C D_\eta^\alpha \zeta(\eta) &= \mathscr{A}\zeta(\eta) + \mathscr{B}\mathbf{u}(\eta) + E\mathbf{w}(\eta) + \sum_{k=1}^{m} A_k \zeta(\eta - d_k(\eta)) + \kappa, \\ y(\eta) &= \mathscr{C}\zeta(\eta) + \mathscr{D}\mathbf{u}(\eta), \qquad \zeta(0) = \zeta_0, \end{cases} \tag{1}$$

where ${}^C D_\eta^\alpha$ is the Caputo fractional derivative (CFD) of order α, $\alpha \in (0,1)$, $\eta \geq 0$ and $\zeta(\eta) \in R^n$ is the state vector, $\mathbf{u}(\eta) \in R^m$ is the control vector, $\mathbf{w}(\eta) \in R^m$ is the disturbance vector [57] for all $\eta \geq 0, \mathscr{A}, A_k, E, \mathscr{B}, \mathscr{C}$, and $\mathscr{D}$ are some constant matrices with appropriate dimensions, $\zeta_0 \in \mathbb{C}([-d_{km}, 0], R^n)$ is an admissible initial state, $\mathbb{C}([-d_{km}, 0], R^n)$ is a Banach space of continuous function $\varphi : [-d_{km}, 0] \to R^n$ and $d_k(\eta)$ is time-varying delay and

$$\kappa = Y\sigma\sqrt{\pi a}, \tag{2}$$

is the SIF where $\kappa \in R^{n\times r}$, Y is the dimensionless constant that depends on the geometry of the cracked body, σ is characteristic stress, and a is the characteristic crack dimension [3].

Here,

$${}^C D_\eta^\alpha \zeta(\eta) = \mathscr{A}\zeta(\eta) + \mathscr{B}\mathbf{u}(\eta) + E\mathbf{w}(\eta) + \sum_{k=1}^{m} A_k \zeta(\eta - d_k(\eta)) + \kappa, \tag{3}$$

and

$$y(\eta) = \mathscr{C}\zeta(\eta) + \mathscr{D}\mathbf{u}(\eta), \tag{4}$$

are state and output equations respectively with $\zeta(0) = \zeta_0$.

3 Preliminaries

In our work, we will use the following definitions and properties to obtain our results for system (1).

Definition 3.1. (*Caputo Fractional Derivative*). *Consider the function $f : \S^+ \to \S$, then the Caputo fractional derivative (CFD) of order α is given as [45]:*

$$ {}^C D_\eta^\alpha \zeta(\eta) = \frac{1}{\Gamma(g-\alpha)} \int_0^\eta \frac{f^g(s)}{(\eta - s)^{\alpha+1-g}} ds, \quad \alpha \in (g-1, g) \; \forall \; g \in \S, \tag{5} $$

where $\Gamma(.)$ is the Euler's gamma function and $\S$ is a complex plane.

Definition 3.2. (*The Mittag-Leffler function*). *The MLF for two parameters $\alpha, \beta > 0$ and $\eta \in \mathbb{R}^+$ is defined as [54]:*

$$ \mathcal{E}_{\alpha,\beta}(h) = \sum_{p=0}^{\infty} \frac{h^p}{\Gamma(\alpha p + \beta)}, \tag{6} $$

where $h, \alpha, \beta \in \S$. We can also obtain,

$$ \varphi_0(\eta) = \mathcal{E}_{\alpha,1}(\mathscr{A}\eta^\alpha) = \sum_{p=0}^{\infty} \frac{\mathscr{A}^p \eta^{\alpha p}}{\Gamma(\alpha p + 1)}, \tag{7} $$

and

$$ \varphi(\eta) = \eta^{\alpha-1} \mathcal{E}_{\alpha,\alpha}(\mathscr{A}\eta^\alpha) = \eta^{\alpha-1} \sum_{p=0}^{\infty} \frac{\mathscr{A}^p \eta^{\alpha p}}{\Gamma(\alpha(p+1))}, \tag{8} $$

from [54] where $\beta = 1$.

The pseudo-transition matrix $\varphi_0(\eta)$ is also known as matrix $\alpha-$ exponential function which can be denoted as $\varphi_0(\eta) = e_\alpha^{\mathscr{A}\eta}$ and for $\alpha = 1$ [32, 38], we have from (7) and (8),

$$ \varphi_0(\eta) = \varphi(\eta) = \sum_{p=0}^{\infty} \frac{\mathscr{A}\eta^p}{\Gamma(p+1)} = e^{\mathscr{A}\eta}. \tag{9} $$

3.1 Some properties of Laplace transform

Here, we explore some key properties of the Laplace transform (LT), which were valuable in finding the solution of fractional-order time-varying delay system (FTVDS) (1).

Some of the famous results are:

- If $0 < \alpha < 1$, then,

$$ \mathcal{L}[(D^\alpha f)(\eta))] = s^\alpha \mathcal{L}[f(\eta)] - s^{\alpha-1} f(0). \tag{10} $$

- Let § be the complex plane, for any $\alpha, \beta > 0$ and $\mathscr{A} \in §^{n\times n}$, then,

$$\mathcal{L}[\eta^{\beta-1}E_{\alpha,\beta}(\mathscr{A}\eta^{\alpha})] = s^{\alpha-\beta}(s^{\alpha}I - \mathscr{A})^{-1}, §(s) > \|\mathscr{A}\| \frac{1}{\alpha}, \tag{11}$$

where $§(s)$ denotes the real component of the complex number [37] and I is identity matrix.

Some formulas for the inverse LT are given as:

$$\mathcal{L}^{-1}(s^{\alpha-1}(s^{\alpha}I - \mathscr{A})^{-1}) = \varphi_0(\eta), \tag{12}$$

$$\mathcal{L}^{-1}((s^{\alpha}I - \mathscr{A})^{-1}) = \varphi(\eta). \tag{13}$$

3.2 Controllability and observability

Both are essential properties while studying dynamical systems [4, 39]. The properties of the controllability Gramian include:

- For every $\eta_f > 0$, the controllability Gramian is symmetric and positive semi-definite.
- The controllability Gramian is positive definite [57] if and only if the state equation is controllable on $[0, \eta_f]$.

The properties of the observability Gramian matrix include:

- For every $\eta_f > 0$, the observability Gramian is symmetric and positive semi-definite.
- The observability Gramian is positive definite [57] if and only if the state equation is observable on $[0, \eta_f]$.

Definition 3.3. (*Controllability*). *System* (1) *is said to be controllable* [9, 71] *on the time interval* $[0, \eta_f], \eta_f > 0$ *whenever* $\exists$ *input signal* $u(.) : [0, \eta_f] \to R^m$ *with the property that the associated solution of system* (1) *fulfills the condition* $\zeta(0) = \zeta_0$ *and* $\zeta(\eta_f) = 0$.

Definition 3.4. (*Observability*). *On time interval* $[0, \eta_f]$, *system* (1) *is said to be observable* [9, 71] *whenever any initial condition (I.C)* $\zeta(0) = \zeta_0 \in R^n$, *for any* $\eta \in [0, \eta_f]$ *and* $\eta_f \in [0, \boldsymbol{\eta}]$, *is lead by the associated system input* $\mathbf{u}(\eta)$ *and output* $y(\eta)$.

Definition 3.5. (*Finite-Time Stability*). *System* (1) *satisfying I.C* $\zeta(\eta) = \Psi_{\zeta}(\eta), -d_{km} \le \eta \le 0$ *is finite stable* [45] *with respect to* $\{\delta, \varepsilon, a_u, a_w, \eta_0, J_0\}$, *where* $\delta < \varepsilon$ *if and only if,*

$$\|\psi_{\zeta}\|_c < \delta,$$

which implies,

$$\|\zeta(\eta)\| < \varepsilon, \ \forall \ \eta \in J_0,$$

where δ *is a real positive number and* $\varepsilon \in R$. *The illustration of the preceding definition is illustrated in* [45].

4 Solution of Fractional-Order Time-Varying Delay System

The solution of state equation (3) on the interval $[0, \eta]$ with I.C $\zeta(0) = \zeta_0$ is

$$\zeta(\eta) = \varphi_0(\eta)\zeta_0 + \int_0^\eta \varphi(\eta - \xi)\left[\mathscr{B}\mathbf{u}(\xi) + E\mathbf{w}(\xi) + \sum_{k=1}^{m} A_k\zeta(\xi - d_k(\xi)) + \kappa\right] d\xi, \tag{14}$$

and the output equation (4) for $\mathscr{D} = 0$ can be written as

$$y(\eta) = \mathscr{C}\left(\varphi_0(\eta)\zeta_0 + \int_0^\eta \varphi(\eta - \xi)\left[\mathscr{B}\mathbf{u}(\xi) + E\mathbf{w}(\xi) + \sum_{k=1}^{m} A_k\zeta(\xi - d_k(\xi)) + \kappa\right] d\xi\right), \tag{15}$$

using the well-known Laplace Transform [1, 32].

5 Finite-Time Stability Analysis of FTVDS

The FTS analysis of FTVDS presented in this paper provides a valuable contribution to the understanding and control of complex dynamical systems, with implications for a wide range of applications in engineering and science.

In the theorem below, we will discuss the FTS analysis of FTVDS.

Theorem 5.1. *The system defined in (1) is finite time stable with respect to $\{\delta, \varepsilon, a_u, a_w, \eta_0, J_0\}$, if the condition below is satisfied, that is*

$$v\left(1 + \frac{\sigma_{\max^*}\eta^\alpha}{\Gamma(\alpha+1)}\right)E_\alpha(v\sigma_{\max^*}\eta^\alpha) + \frac{v\gamma\eta^\alpha}{\Gamma(\alpha+1)} \leq \frac{\varepsilon}{\delta},\ \forall\ \eta \in J_0 = [0, \boldsymbol{\eta}], \tag{16}$$

where $v = \left(1 + \frac{\sigma_{\mathscr{A}}\eta^\alpha}{\Gamma(\alpha+1)}\right)^{-1}$, $\delta < \varepsilon$ and system (1) satisfies I.C $\zeta(\eta) = \Psi_\zeta(\eta), -d_{km} \leq \eta \leq 0$.

In (16), $\sigma_{\max}^{(.)}$ is the largest singular value (SV) of matrix (.) and is defined as $\sigma_{\max^*} = \sigma_{\max}(\mathscr{A}) + \sigma_{\max}\left(\sum_{k=1}^{m} A_k\right)$. Also,

$$\gamma = \frac{ba_u + ea_w + ķ}{\delta}, \tag{17}$$

where $b = \|\mathscr{B}\|$, $e = \|E\|$, $ķ = \|\kappa\|$, $a_u = \|\mathbf{u}(\eta)\|$ and $a_w = \|\mathbf{w}(\eta)\|$.

Proof. Applying norm $\|.\|$ on (14) and from Definition 3.2 for $p = 0$ and $\eta_0 = 0$, we have

$$\|\zeta(\eta)\| \leq \|\zeta_0\| + \frac{1}{\Gamma(\alpha)}\int_0^\eta \left|(\eta - \xi)^{\alpha-1}\right| \left\|\mathscr{B}\mathbf{u}(\xi). + E\mathbf{w}(\xi) + \sum_{k=1}^{m} A_k\zeta(\xi - d_k(\xi)) + \kappa\right\| d\xi, \tag{18}$$

also applying norm $\|.\|$ on (3), we have

$$\left\|{}^C D_\eta^\alpha\zeta(\eta)\right\| \leq \left\|\mathscr{A}\zeta(\eta) + \mathscr{B}\mathbf{u}(\eta) + E\mathbf{w}(\eta) + \sum_{k=1}^{m} A_k\zeta(\eta - d_k(\eta)) + \kappa\right\|, \tag{19}$$

$$\left\|{}^{C}D_{\eta}^{\alpha}\zeta(\eta)\right\| \leq \|\mathscr{A}\|\,\|\zeta(\eta)\| + \|\mathscr{B}\|\,\|\mathbf{u}(\eta)\| + \|E\|\,\|\mathbf{w}(\eta)\| + \sum_{k=1}^{m}\|A_k\|\,\|\zeta(\eta-d_k(\eta))\| + \|\kappa\|, \tag{20}$$

and we can write (20) as

$$\left\|{}^{C}D_{\eta}^{\alpha}\zeta(\eta)\right\| \leq (\sigma_{\max}(\mathscr{A}))\,\|\zeta(\eta)\| + b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| + \left(\sigma_{\max}\left(\sum_{k=1}^{m}A_k\right)\right)\|\zeta(\eta-d_k(\eta))\| + ķ, \tag{21}$$

where $\|\mathscr{A}\|$ =maximum SV of $\mathscr{A} = \sigma_{\max}(\mathscr{A})$, $\|A_k\|$ =maximum SV of $A_k = \sigma_{\max}\left(\sum_{k=1}^{m}A_k\right)$ [44, 45]. Also from [43], we have,

$$\|\zeta(\eta-d_k(\eta))\| \leq \sup_{\eta-d_{km}\leq\eta^{\cdot}\leq\eta}\|\zeta(\eta^{\cdot})\|, \qquad \eta^{\cdot}\in[\eta-d_{km},\eta], \tag{22}$$

where d_{km} is some k^{th} scalar, i.e. $k=1,2,\ldots,m$.

From (22), (21) becomes,

$$\begin{aligned}\left\|{}^{C}D_{\eta}^{\alpha}\zeta(\eta)\right\| &\leq \sigma_{\max}(\mathscr{A})\,\|\zeta(\eta)\| + b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| \\ &\quad + \sigma_{\max}\left(\sum_{k=1}^{m}A_k\right)\sup_{\eta-d_{km}\leq\eta^{\cdot}\leq\eta}\|\zeta(\eta^{\cdot})\| + ķ,\end{aligned} \tag{23}$$

$$\left\|{}^{C}D_{\eta}^{\alpha}\zeta(\eta)\right\| \leq \sigma_{\max^*}\sup_{\eta-d_{km}\leq\eta^{\cdot}\leq\eta}\|\zeta(\eta^{\cdot})\| + b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| + ķ, \quad \eta > d_{km}, \tag{24}$$

where

$$\sigma_{\max^*} = \sigma_{\max}(\mathscr{A}) + \sigma_{\max}\left(\sum_{k=1}^{m}A_k\right), \tag{25}$$

for $k=1,2,\ldots,m$ and from (19) and (24) for $\eta_{0+}\in\eta$, we can write,

$$\begin{aligned}&\left\|\mathscr{A}\zeta(\eta) + \mathscr{B}\mathbf{u}(\eta) + E\mathbf{w}(\eta) + \sum_{k=1}^{m}A_k\zeta(\eta-d_k(\eta)) + \kappa\right\| \\ &\leq \left(\sigma_{\max^*}\sup_{\eta-d_{km}\leq\eta^{\cdot}\leq\eta}\|\zeta(\eta^{\cdot})\| + \|\psi_{\zeta}\|_c\right) + b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| + ķ, \eta > \eta_{0+},\end{aligned} \tag{26}$$

and in view of (26), for $k=1,2,\ldots,m$, (18) becomes,

$$\begin{aligned}\|\zeta(\eta)\| &\leq \|\zeta_0\| + \frac{1}{\Gamma(\alpha)}\int_0^{\eta}\left|(\eta-\xi)^{\alpha-1}\right|\left(\sigma_{\max^*}\left(\sup_{\eta-d_{km}\leq\eta^{\cdot}\leq\eta}\|\zeta(\eta^{\cdot})\| + \|\psi_{\zeta}\|_c\right)\right. \\ &\qquad \left. + b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| + ķ - \|\mathscr{A}\zeta(\eta)\|\right)d\xi \\ &\leq \|\zeta_0\| + \frac{1}{\Gamma(\alpha)}\int_0^{\eta}\left|(\eta-\xi)^{\alpha-1}\right|\left(\sigma_{\max^*}\left(\sup_{\eta-d_{km}\leq\eta^{\cdot}\leq\eta}\|\zeta(\eta^{\cdot})\| + \|\psi_{\zeta}\|_c\right)\right. \\ &\qquad \left. + b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| + ķ - \sigma_{\mathscr{A}}\,\|\zeta(\eta)\|\right)d\xi,\end{aligned}$$

where $\sigma_{\mathscr{A}} = \sigma_{\max}(\mathscr{A})$ [44]. Now,

$$\le \|\zeta_0\| + \frac{1}{\Gamma(\alpha)} \int_0^\eta \left|(\eta-\xi)^{\alpha-1}\right| \left(\sigma_{\max^*}\left(\sup_{\eta-d_{km}\le\eta^{\cdot}\le\eta} \|\zeta(\eta^{\cdot})\| + \|\psi_\zeta\|_c\right)\right) d\xi$$
$$+ \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left(b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| + \c{k} - \sigma_{\mathscr{A}}\,\|\zeta(\eta)\|\right)$$
$$\le \|\zeta_0\| + \frac{1}{\Gamma(\alpha)} \int_0^\eta \left|(\eta-\xi)^{\alpha-1}\right| \left(\sigma_{\max^*}\left(\sup_{\eta-d_{km}\le\eta^{\cdot}\le\eta} \|\zeta(\eta^{\cdot})\| + \|\zeta_0\|\right)\right) d\xi$$
$$+ \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left(b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| + \c{k} - \sigma_{\mathscr{A}}\,\|\zeta(\eta)\|\right)$$
$$\le \left(1 + \frac{\sigma_{\max^*}\eta^\alpha}{\Gamma(\alpha+1)}\right)\|\zeta_0\| + \frac{\sigma_{\max^*}}{\Gamma(\alpha)} \int_0^\eta \left|(\eta-\xi)^{\alpha-1}\right| \left(\sup_{\eta-d_{km}\le\eta^{\cdot}\le\eta} \|\zeta(\eta^{\cdot}\|\right) d\xi$$
$$+ \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left(b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| + \c{k} - \sigma_{\mathscr{A}}\,\|\zeta(\eta)\|\right), \tag{27}$$

or

$$\left(1 + \frac{\sigma_{\mathscr{A}}\eta^\alpha}{\Gamma(\alpha+1)}\right)\|\zeta(\eta)\| \le \left(1 + \frac{\sigma_{\max^*}\eta^\alpha}{\Gamma(\alpha+1)}\right)\|\zeta_0\| + \frac{\sigma_{\max^*}}{\Gamma(\alpha)} \int_0^\eta \left|(\eta-\xi)^{\alpha-1}\right|$$
$$\times \left(\sup_{\eta-d_{km}\le\eta^{\cdot}\le\eta} \|\zeta(\eta^{\cdot})\|\right) d\xi + \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left(b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| + \c{k}\right)$$
$$\|\zeta(\eta)\| \le \left(1 + \frac{\sigma_{\mathscr{A}}\eta^\alpha}{\Gamma(\alpha+1)}\right)^{-1} \left[\left(1 + \frac{\sigma_{\max^*}\eta^\alpha}{\Gamma(\alpha+1)}\right)\|\zeta_0\| + \frac{\sigma_{\max^*}}{\Gamma(\alpha)} \int_0^\eta \left|(\eta-\xi)^{\alpha-1}\right|\right.$$
$$\left. \times \left(\sup_{\eta-d_{km}\le\eta^{\cdot}\le\eta} \|\zeta(\eta^{\cdot})\|\right) d\xi + \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left(b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| + \c{k}\right)\right]. \tag{28}$$

Consider $\left(1 + \frac{\sigma_{\mathscr{A}}\eta^\alpha}{\Gamma(\alpha+1)}\right)^{-1} = v$, then (28) can be written in the form,

$$\|\zeta(\eta)\| \le v\left(1 + \frac{\sigma_{\max^*}\eta^\alpha}{\Gamma(\alpha+1)}\right)\|\zeta_0\| + v\frac{\sigma_{\max^*}}{\Gamma(\alpha)} \int_0^\eta \left|(\eta-\xi)^{\alpha-1}\right|$$
$$\times \left(\sup_{\eta-d_{km}\le\eta^{\cdot}\le\eta} \|\zeta(\eta^{\cdot})\|\right) d\xi + \frac{v\eta^\alpha}{\Gamma(\alpha+1)} \left(b\,\|\mathbf{u}(\eta)\| + e\,\|\mathbf{w}(\eta)\| + \c{k}\right). \tag{29}$$

From the generalized Gronwall inequality, which states that:

For some $\boldsymbol{\eta} \le +\infty$, consider two non-negative functions $\zeta(\eta)$ and $\mathbf{a}(\boldsymbol{\eta})$ which are locally integrable on $0 \le \eta < \boldsymbol{\eta}$. Let $\mathbf{g}(\boldsymbol{\eta})$ be a continuous function that is nonnegative, non-decreasing, and is defined on $0 \le \eta < \boldsymbol{\eta}$ where $\mathbf{g}(\boldsymbol{\eta}) \le M$ =constant and $\alpha > 0$ with,

$$\zeta(\eta) \le \mathbf{a}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta}) \int_0^\eta (\eta-s)^{\alpha-1}\zeta(s)ds, \tag{30}$$

on this interval [11]. Then,

$$\zeta(\eta) \leq \mathbf{a}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta}) \int_0^{\eta} \left[\sum_{n=1}^{\infty} \frac{(\mathbf{g}(\boldsymbol{\eta})\Gamma(\alpha))^n}{\Gamma(n\alpha)} (\eta - s)^{n\alpha-1} \mathbf{a}(\mathbf{s}) \right] ds, \quad 0 \leq \eta < \boldsymbol{\eta}, \tag{31}$$

and from the corollary in [44], which states that:

On interval $[0, \boldsymbol{\eta})$, let $\mathbf{a}(\boldsymbol{\eta})$ be a function which is non-decreasing. Then, the following inequality holds:

$$\zeta(\eta) \leq \mathbf{a}(\boldsymbol{\eta}) E_\alpha(\mathbf{g}(\boldsymbol{\eta})\Gamma(\alpha)\eta^\alpha). \tag{32}$$

Here, E_α is MLF.

Then, from (29) we have $\mathbf{g}(\boldsymbol{\eta}) = v \left(\frac{\sigma_{\max^*}}{\Gamma(\alpha)} \right)$ and one can introduce a non-decreasing function [44], $\mathbf{a}(\boldsymbol{\eta})$ such as

$$\mathbf{a}(\boldsymbol{\eta}) = v \left(1 + \frac{\sigma_{\max^*} \eta^\alpha}{\Gamma(\alpha+1)} \right) \|\zeta_0\| \,. \tag{33}$$

In view of (30), (29) becomes,

$$\begin{aligned} \|\zeta(\eta)\| \leq \mathbf{a}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta}) \int_0^{\eta} \left|(\eta - \xi)^{\alpha-1}\right| \left(\sup_{\eta - d_{km} \leq \eta^{\cdot} \leq \eta} \|\zeta(\eta^{\cdot})\| \right) d\xi \\ + \frac{v\eta^\alpha}{\Gamma(\alpha+1)} \left(b\|\mathbf{u}(\eta)\| + e\|\mathbf{w}(\eta)\| + \c{k} \right). \end{aligned} \tag{34}$$

Let us define from (34),

$$\left\|\zeta^\dagger(\eta)\right\| = \mathbf{a}(\boldsymbol{\eta}) + \mathbf{g}(\boldsymbol{\eta}) \int_0^{\eta} \left|(\eta - \xi)^{\alpha-1}\right| \left(\sup_{\eta - d_{km} \leq \eta^{\cdot} \leq \eta} \|\zeta(\eta^{\cdot})\| \right) d\xi. \tag{35}$$

Then, we can write (34) as

$$\|\zeta(\eta)\| \leq \left\|\zeta^\dagger(\eta)\right\| + \frac{v\eta^\alpha}{\Gamma(\alpha+1)} \left(b\|\mathbf{u}(\eta)\| + e\|\mathbf{w}(\eta)\| + \c{k} \right), \tag{36}$$

and it is easy to show from (32) and (35),

$$\begin{aligned} \left\|\zeta^\dagger(\eta)\right\| &\leq \mathbf{a}(\boldsymbol{\eta}) E_\alpha(\mathbf{g}(\boldsymbol{\eta})\Gamma(\alpha)\eta^\alpha) \\ &= v \left(1 + \frac{\sigma_{\max^*}\eta^\alpha}{\Gamma(\alpha+1)} \right) \times \|\zeta_0\| E_\alpha \left(v \frac{\sigma_{\max^*}}{\Gamma(\alpha)} \Gamma(\alpha)\eta^\alpha \right). \end{aligned} \tag{37}$$

So, (36) becomes,

$$\|\zeta(\eta)\| \leq v \left(1 + \frac{\sigma_{\max^*}\eta^\alpha}{\Gamma(\alpha+1)} \right) \delta E_\alpha \left(v\sigma_{\max^*}\eta^\alpha \right) + \frac{v\eta^\alpha}{\Gamma(\alpha+1)} \left(b\|\mathbf{u}(\eta)\| + e\|\mathbf{w}(\eta)\| + \c{k} \right). \tag{38}$$

For simplicity, let us write from (38),

$$\delta\gamma = \left(ba_u + ea_w + \c{k} \right). \tag{39}$$

Now, from (38), we get

$$\|\zeta(\eta)\| \leq \delta v \left(1 + \frac{\sigma_{\max^*}\eta^\alpha}{\Gamma(\alpha+1)}\right) E_\alpha\left(v\sigma_{\max^*}\eta^\alpha\right) + \frac{v\delta\gamma\eta^\alpha}{\Gamma(\alpha+1)}, \tag{40}$$

and by using the condition in Theorem 5.1, the relationship in (16) gives,

$$\|\zeta(\eta)\| < \varepsilon, \ \forall \ \ \eta \in J_0. \tag{41}$$

This completes the proof. □

6 Controllability of FTVDS

The necessary and sufficient conditions of controllability for the FTVDS defined in system (1) are explored in this section.

Theorem 6.1. *The system defined in (1) is controllable on* $[0, \eta_f]$ *iff,*

$$W_c[0,\eta_f]_{n\times n} = \int_0^{\eta_f} \varphi\left(\eta_f - \xi\right) \mathscr{B}\mathscr{B}^* \varphi^*\left(\eta_f - \xi\right) d\xi, \tag{42}$$

the Gramian controllability matrix is non-singular. Here $*$ *represents the matrix transpose and* $\varphi^*\left(\eta\right) = E_{\alpha,\alpha}(\mathscr{A}^*\eta^\alpha)$*,* $\eta_f > 0$*.*

Proof. Let us define a control input $\mathbf{u}(\eta)$ for our FTVDS in system (1) as

$$\begin{aligned}\mathbf{u}(\eta) = \mathscr{B}^*\varphi^*\left(\eta-\xi\right) W_c^{-1}[0,\eta]\Bigg(&-\varphi_0(\eta)\zeta(0) - \int_0^{\eta} \varphi\left(\eta-\xi\right) \\ &\times \left[\mathscr{B}\mathbf{u}(\xi) + E\mathbf{w}(\xi) + \sum_{k=1}^{m} A_k\zeta(\xi - d_k(\xi)) + \kappa\right] d\xi\Bigg).\end{aligned} \tag{43}$$

Substituting the control input in (14) for $\eta = \eta_f$ we get

$$\begin{aligned}\zeta(\eta_f) = \varphi_0(\eta_f)\zeta_0 + &\int_0^{\eta_f} \varphi(\eta_f - \xi)\Bigg[\mathscr{B}(\mathscr{B}^*\varphi^*\left(\eta_f-\xi\right) W_c^{-1}[0,\eta_f] \\ &\times \left(-\varphi_0(\eta_f)\zeta(0) - \int_0^{\eta_f} \varphi\left(\eta_f-\xi\right) \left[E\mathbf{w}(\xi) + \sum_{k=1}^{m} A_k\zeta(\xi - d_k(\xi))) + \kappa\right]\right) \\ &+ E\mathbf{w}(\xi) + \sum_{k=1}^{m} A_k\zeta(\xi - d_k(\xi)) + \kappa\Bigg] d\xi,\end{aligned} \tag{44}$$

which implies that,

$$\zeta(\eta_f) = 0. \tag{45}$$

Alternatively, with no loss of generality, if $W_c[0,\eta_f]$ is singular [71], $\exists\ z \in \mathbb{R}^n$ vector which is nonzero s.t. $z^* W_c[0,\eta_f] z = 0$. This gives

$$z^* \int_0^{\eta_f} \varphi\left(\eta_f-\xi\right) \mathscr{B}\mathscr{B}^* \varphi^*\left(\eta_f-\xi\right) z d\xi = 0. \tag{46}$$

It gives

$$z^* \varphi\left(\eta_f-\xi\right) \mathscr{B} = 0. \tag{47}$$

We consider $\zeta_0 = \varphi_0^{-1}(\eta_f) z$. Given the assumption over the interval $[0,\eta_f]$, $\exists$ input u which drives ζ_0 to the origin. It follows from (14) as

$$\begin{aligned} \zeta(\eta_f) &= \varphi_0(\eta_f)\varphi_0^{-1}(\eta_f) z + \int_0^{\eta_f} \varphi(\eta_f-\xi)\left[\mathscr{B}\mathbf{u}(\xi) + E\mathbf{w}(\xi) + \sum_{k=1}^{m} A_k \zeta(\xi - d_k(\xi)) + \kappa\right] d\xi \\ &= z + \int_0^{\eta_f} \varphi(\eta_f-\xi)\left[\mathscr{B}\mathbf{u}(\xi) + E\mathbf{w}(\xi) + \sum_{k=1}^{m} A_k \zeta(\xi - d_k(\xi)) + \kappa\right] d\xi = 0. \end{aligned} \tag{48}$$

Then,

$$z^* z + \int_0^{\eta_f} z^* \varphi(\eta_f-\xi)\left[\mathscr{B}\mathbf{u}(\xi) + E\mathbf{w}(\xi) + \sum_{k=1}^{m} A_k \zeta(\xi - d_k(\xi)) + \kappa\right] d\xi = 0. \tag{49}$$

The second term is zero here, so it concludes that $z^* z = 0$. This is a contradiction to $z \neq 0$. Therefore, $W_c[0,\eta_f]$ is non-singular. Hence, proved. □

Now we will discuss the rank condition for the controllability in system (1).

Theorem 6.2. *The system in (1) is controllable on $[0,\eta_f]$ iff,*

$$rank(Q_c) = n, \tag{50}$$

where

$$\begin{aligned} Q_c = [&\mathscr{B}|\mathscr{A}\mathscr{B}|\ldots|\mathscr{A}^{n-1}\mathscr{B}|E|\mathscr{A}E|\ldots|\mathscr{A}^{n-1}E| \\ &\sum_{k=1}^{m} A_k|\mathscr{A}\sum_{k=1}^{m} A_k|\ldots|\mathscr{A}^{n-1}\sum_{k=1}^{m} A_k|\kappa|\mathscr{A}\kappa|\ldots|\mathscr{A}^{n-1}\kappa], \end{aligned} \tag{51}$$

and the order of Q_c is $n \times n$.

Proof. From Cayley-Hamilton's Theorem [9, 71],

$$\varphi(\eta-\xi) = \sum_{i=0}^{n-1} r_i(\eta-\xi)\mathscr{A}^i. \tag{52}$$

Here, $r_i(\eta)$ is the polynomial in η.

Using (52) in (14) and from Definition 3.2, we get

$$\zeta(\eta) - E_{\alpha,1}\left(\mathscr{A}\eta^{\alpha}\right)\zeta(0) = \sum_{i=0}^{n-1}\mathscr{A}^i\mathscr{B}\int_0^{\eta_f} r_i(\eta_f-\xi)\mathbf{u}(\xi)d\xi + \sum_{i=0}^{n-1}\mathscr{A}^iE\int_0^{\eta_f} r_i(\eta_f-\xi)\mathbf{w}(\xi)d\xi$$
$$+\sum_{i=0}^{n-1}\sum_{k=1}^{m}\mathscr{A}^iA_k\int_0^{\eta_f} r_i(\eta_f-\xi)\zeta(\xi-d_k(\xi))d\xi + \sum_{i=0}^{n-1}\mathscr{A}^i\kappa\int_0^{\eta_f} r_i(\eta_f-\xi)d\xi. \tag{53}$$

Then, from (53) we have

$$\zeta(\eta) - E_{\alpha,1}\left(\mathscr{A}\eta^{\alpha}\right)\zeta(0) = [\mathscr{B}|\mathscr{A}\mathscr{B}|\dots|\mathscr{A}^{n-1}\mathscr{B}|E|\mathscr{A}E|\dots|\mathscr{A}^{n-1}E|$$
$$\sum_{k=1}^{m}A_k|\mathscr{A}\sum_{k=1}^{m}A_k|\dots|\mathscr{A}^{n-1}\sum_{k=1}^{m}A_k|\kappa|\mathscr{A}\kappa\dots|\mathscr{A}^{n-1}\kappa]\times\begin{bmatrix} l_0\\ l_1\\ \vdots\\ l_{n-1}\\ m_0\\ m_1\\ \vdots\\ m_{n-1}\\ n_0\\ n_1\\ \vdots\\ n_{n-1}\\ p_0\\ p_1\\ \vdots\\ p_{n-1}\end{bmatrix}, \tag{54}$$

where

$$l_i = \sum_{i=0}^{n-1}\int_0^{\eta_f} r_i(\eta_f-\xi)\mathbf{u}(\xi)d\xi, \qquad m_i = \sum_{i=0}^{n-1}\int_0^{\eta_f} r_i(\eta_f-\xi)\mathbf{w}(\xi)d\xi,$$

$$n_i = \sum_{i=0}^{n-1}\sum_{k=1}^{m}\int_0^{\eta_f} r_i(\eta_f-\xi)\zeta(\xi-d_k(\xi))d\xi, \qquad p_i = \sum_{i=0}^{n-1}\int_0^{\eta_f} r_i(\eta_f-\xi)d\xi.$$

Hence, proved. □

7 Observability of FTVDS

Theorem 7.1. *System (1) is observable on interval $[0,\eta_f]$ iff $n\times n$ observability Gramian matrix,*

$$W_o[0,\eta_f]_{n\times n} = \int_0^{\eta_f}\varphi_0^*(\eta)\mathscr{C}^*\mathscr{C}\varphi_0(\eta)\,d\eta, \tag{55}$$

is non-singular. Here $*$ represents the matrix transpose and $\varphi_0^*(\eta) = E_{\alpha,\beta}(\mathscr{A}^*\eta^\alpha)$. The matrix $W_o[0,\eta_f]_{n\times n}$ which is Gramian matrix is invertible and $\eta_f \in [0,\boldsymbol{\eta}]$ is the time interval over which the system's behavior is defined.

Proof. Let us define $\overline{y}(\eta)$ in view of (4) as

$$\overline{y}(\eta) = y(\eta) - \mathscr{C}\left(\varphi_0(\eta)\zeta_0 + \int_0^\eta \varphi(\eta-\xi)\left[\mathscr{B}\mathbf{u}(\xi) + E\mathbf{w}(\xi) + \sum_{k=1}^m A_k\zeta(\xi - d_k(\xi)) + \kappa\right]d\xi\right). \tag{56}$$

Then,

$$\overline{y}(\eta) = \mathscr{C}\varphi_0(\eta)\zeta_0. \tag{57}$$

The estimation of ζ_0 from $y(\eta)$ is equivalent to the system's observability in (1). Since ζ_0 and $\overline{y}(\eta)$ are arbitrary, this leads to the estimation of ζ_0 from $y(\eta)$, which, for $\mathbf{u}(\eta) = 0$, is given by

$$y(\eta) = \mathscr{C}\varphi_0(\eta)\zeta_0. \tag{58}$$

$W_o^{-1}[0,\eta_f]$ is well defined if $W_o[0,\eta_f]$ is non-singular. The following expression for the arbitrary $y(\eta)$ for $\eta_f > 0$ is given as:

$$W_o^{-1}[0,\eta_f]\int_0^{\eta_f}\varphi_0^*(\eta)\,\mathscr{C}^* y(\eta)\,d\eta = W_o^{-1}[0,\eta_f]\int_0^{\eta_f}\varphi_0^*(\eta)\,\mathscr{C}^*\mathscr{C}\varphi_0(\eta)\zeta_0 d\eta, \tag{59}$$

$$W_o^{-1}[0,\eta_f]\int_0^{\eta_f}\varphi_0^*(\eta)\,\mathscr{C}^*\mathscr{C}\varphi_0(\eta)\zeta_0 d\eta = W_o^{-1}[0,\eta_f]W_o[0,\eta_f]\zeta_0. \tag{60}$$

It follows that,

$$W_o^{-1}[0,\eta_f]\int_0^{\eta_f}\varphi_0^*(\eta)\,\mathscr{C}^* y(\eta)\,d\eta = \zeta_0. \tag{61}$$

The left hand side of (61) is a linear algebraic equation of ζ_0 and depends on $y(\eta) \in [0,\eta_f]$. As $W_o[0,\eta_f]$ is invertible for $\eta \in [0,\eta_f]$, the initial state $\zeta(0) = \zeta_0$ can be uniquely inferred from the system output $y(\eta)$. Conversely, if $W_o[0,\eta_f]$ is singular for $\eta_f > 0$, then a vector ζ_α which is non zero exists such that,

$$\zeta_\alpha^* W_o[0,\eta_f]\zeta_\alpha = 0. \tag{62}$$

Choosing $\zeta_\alpha = \zeta_0$, then we get

$$\int_0^{\eta_f} y^*(\eta)y(\eta)\,d\eta = \zeta_0^*\left(\int_0^{\eta_f}\varphi_0^*(\eta)\,\mathscr{C}^*\mathscr{C}\varphi_0(\eta)\,d\eta\right)\zeta_0, \tag{63}$$

which gives

$$\int_0^{\eta_f}\|y(\eta)\|^2\,d\eta = 0. \tag{64}$$

Therefore,

$$y(\eta) = \mathscr{C}\varphi_0(\eta)\zeta_0. \tag{65}$$

Given that the system is observable which follows $\zeta_0 = 0$, leading to a contradiction. So, $W_o[0,\eta_f]$ is non-singular. This completes the proof. □

Now we will discuss the rank condition for the observability in system (1).

Theorem 7.2. *The system given in* (1) *is observable on* $[0, \eta_f]$ *iff,*

$$rank(Q_o) = n, \tag{66}$$

where

$$Q_o = \begin{bmatrix} \mathscr{C} \\ \mathscr{C}\mathscr{A} \\ \vdots \\ \mathscr{C}\mathscr{A}^{n-1} \end{bmatrix}, \tag{67}$$

and the order of Q_o *is* $n \times n$.

Proof. Theorem 7.1 gives

$$y(\eta) = \mathscr{C}\varphi_0(\eta)\zeta_0, \tag{68}$$

and ζ_0 is uniquely determined by $y(\eta)$ iff $\mathscr{C}\varphi_0(\eta)$ is non-singular [9]. Using Cayley-Hamilton theorem [9, 71],

$$\mathscr{C}\varphi_0(\eta) = \mathscr{C}\sum_{j=0}^{n-1} b_j(\eta)\mathscr{A}^j, \tag{69}$$

where $b_j(\eta)$ is a function that is a polynomial in η. So, it follows for $\eta = \eta_f$,

$$\mathscr{C}\varphi_0(\eta_f) = \sum_{j=0}^{n-1} b_j(\eta_f)\mathscr{C}\mathscr{A}^j, \tag{70}$$

$$\mathscr{C}\varphi_0(\eta_f) = (b_0(\eta_f), b_1(\eta_f), \dots, b_{n-1}(\eta_f)) \begin{bmatrix} \mathscr{C} \\ \mathscr{C}\mathscr{A} \\ \vdots \\ \mathscr{C}\mathscr{A}^{n-1} \end{bmatrix}. \tag{71}$$

Then, $\mathscr{C}\varphi_0(\eta_f)$ is non-singular iff,

$$rank \begin{bmatrix} \mathscr{C} \\ \mathscr{C}\mathscr{A} \\ \vdots \\ \mathscr{C}\mathscr{A}^{n-1} \end{bmatrix} = n, \tag{72}$$

$$rank(Q_o) = n. \tag{73}$$

Therefore, it completes the proof. □

Now, we will elaborate our results using some examples.

8 Examples and Results

Example 8.1. *In this example, we will discuss the finite-time stability of the system,*

$$\begin{cases} {}^{C}D_{\eta}^{\alpha}\zeta(\eta) &= \mathscr{A}\zeta(\eta) + \mathscr{B}\mathbf{u}(\eta) + E\mathbf{w}(\eta) + \sum_{k=1}^{m} A_k\zeta(\eta - d_k(\eta)) + \kappa, \\ y(\eta) &= \mathscr{C}\zeta(\eta), \qquad \eta \in [0,1], \end{cases} \tag{74}$$

using Theorem 5.1, with the following assumptions:

$$\mathscr{A} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, \qquad \mathscr{B} = \begin{bmatrix} 5.9 \\ -1 \end{bmatrix}, \qquad \mathscr{C} = \begin{bmatrix} 4 & 2 \end{bmatrix}, \qquad \zeta_0 = \begin{bmatrix} 0 \\ -0.05 \end{bmatrix},$$
$$E = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & -1 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \kappa = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 6\sqrt{2\pi} \\ 6\sqrt{\pi} \end{bmatrix},$$

where $Y_1 = 3, m = 2, \sigma_1 = 2, a_1 = 2$ *and* $Y_2 = 2, \sigma_2 = 3, a_2 = 1$. *Firstly, we have obtained the solution of system (74) for the considered values, and further, we will discuss the finite-time stability for this system as well.*

For $p = 0$ *and* $\alpha = 0.5$, *we have from Definition 3.2,*

$$\varphi(1-\xi) = \frac{(1-\xi)^{-0.5}}{\Gamma(0.5)}, \qquad \varphi_0(\eta) = I_{2\times 2}.$$

Also, by substituting the values,

$$\zeta(\xi - d_1(\xi)) = \begin{bmatrix} 0.01\xi - \cos(0.01\xi) \\ 0.01\xi - \cos(0.01\xi) \end{bmatrix}, \quad \zeta(\xi - d_2(\xi)) = \begin{bmatrix} 0.1\xi^2 - \cos(0.1\xi^2) \\ 0.1\xi^2 - \cos(0.1\xi^2) \end{bmatrix},$$
$$\mathbf{u}(\eta) = -\eta + 2, \qquad w(\eta) = \begin{bmatrix} -1.5 \\ -\eta \end{bmatrix},$$

in (74), we obtain the approximate solution,

$$\begin{bmatrix} \zeta_1(\eta) \\ \zeta_2(\eta) \end{bmatrix} = \begin{bmatrix} (2.3\sqrt{\eta} - 2.3\sqrt{\eta-1})\cos(0.1\eta^2) + (5.2\eta - 31.1 \\ +1.13\cos(0.01\eta) + 0.23\eta^2)\sqrt{\eta-1} - 5.2\eta^{\frac{3}{2}} \\ +33.7\sqrt{\eta} - 1.13\sqrt{\eta}\cos(0.01\eta) - 0.23\eta^{\frac{5}{2}} \\ (-1.13\sqrt{\eta} + 1.13\sqrt{\eta-1})\cos(0.1\eta^2) + (-1.5\eta - 10.5 \\ -2.3\cos(0.01\eta) - 0.113\eta^2)\sqrt{\eta-1} - 0.05 + 1.5\eta^{\frac{3}{2}} \\ +9.74\sqrt{\eta} + 2.3\sqrt{\eta}\cos(0.01\eta) + 0.113\eta^{\frac{5}{2}} \end{bmatrix}. \tag{75}$$

The graph of the solution of (74) is illustrated in Figure 1 below:

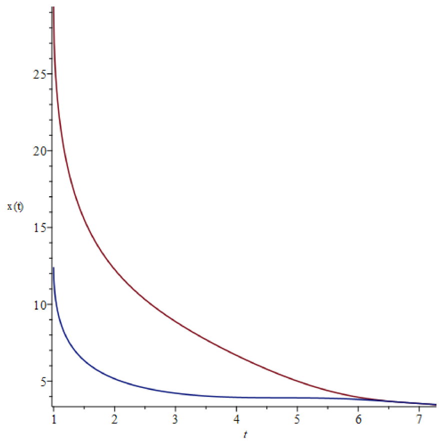

Figure 1: Graphical analysis of the stability of the fractional-order model in (74).

For various values of α, we have obtained the following graphs given in Figure 2:

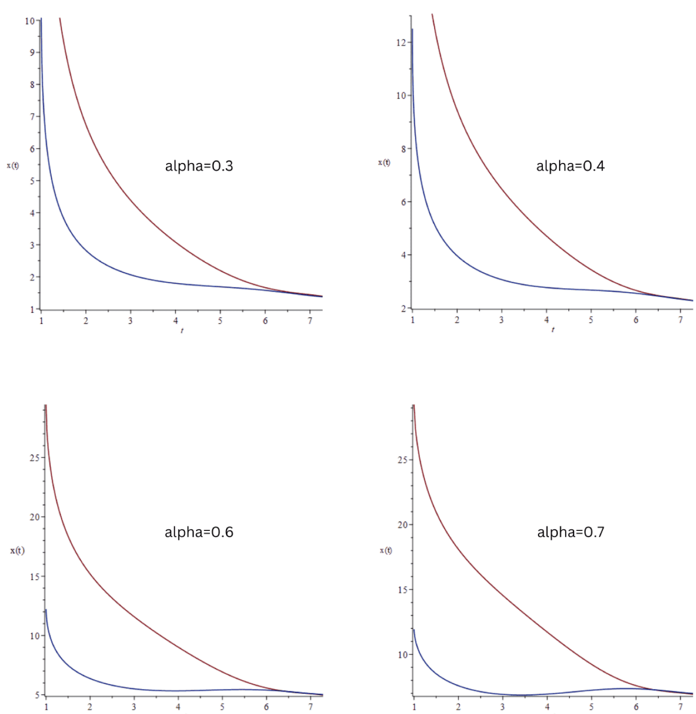

Figure 2: Graphical analysis of the fractional-order model in (74) with different values of α.

The comparison graphs of the stability of the fractional/non-integer order model approaching classical order are shown in Figure 3.

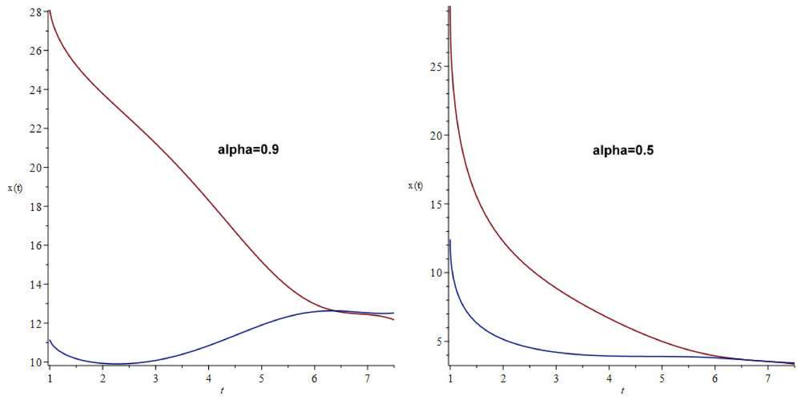

Figure 3: Graphical analysis of the fractional-order model in (74) with different values of fractional-order.

In (74), if the fractional-order derivative is replaced by the classical order derivative, we get

$$\begin{cases} \dfrac{d}{d\eta}\zeta(\eta) &= \mathscr{A}\zeta(\eta) + \mathscr{B}\mathbf{u}(\eta) + E\mathbf{w}(\eta) + \displaystyle\sum_{k=1}^{m} A_k\zeta(\eta - d_k(\eta)) + \kappa, \\ y(\eta) &= \mathscr{C}\zeta(\eta), \qquad \eta \in [0,1]\,. \end{cases} \tag{76}$$

Then, the graph of the solution of (76) is illustrated in Figure 4.

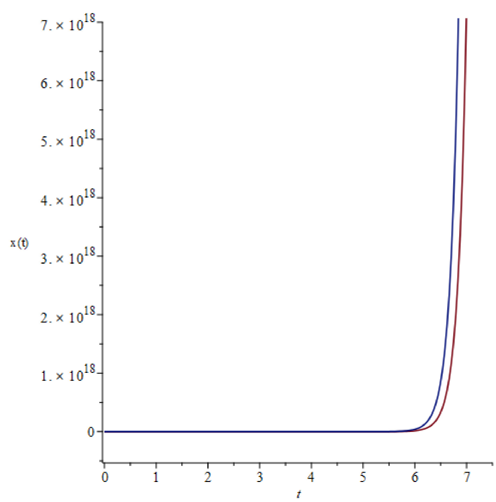

Figure 4: Graphical analysis of the solution of the classical-order model in (76).

It is seen that when system (74) is incorporated with a classical derivative, then we get a clearly unstable solution graph.

The presence of external disturbances or uncertainties affects the robustness of the system's stability. In Figure 5(a), we have $w(\eta) = \begin{bmatrix} 1.4 \\ \eta \end{bmatrix}$ *and in Figure 5(b), we have* $w(\eta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$*. From Figures 5(a) and 5(b), it can be seen that the robustness of the system stability is affected.*

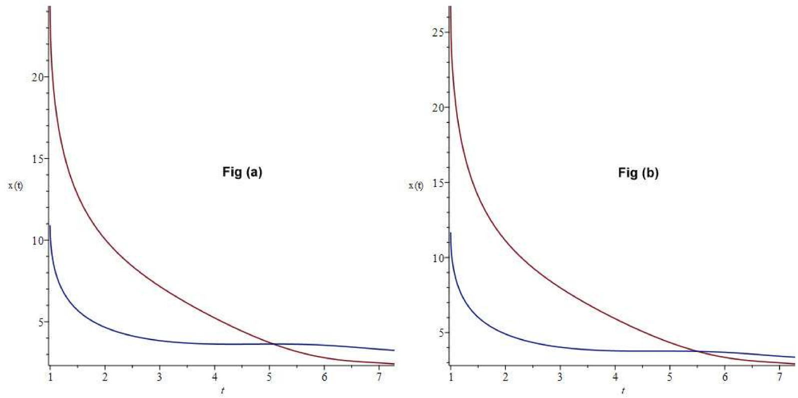

Figure 5: Graphical analysis of the fractional-order model in (74) with different values of external disturbances.

Now we will check the FTS using Theorem 5.1 with respect to

$$\eta_0 = 0, \quad J = \{0, 1\}, \quad \delta = 0.06, \quad \varepsilon = 30, \quad a_u = 1, \quad a_w = 1.5, \quad p = 0,$$
$$\text{for } -1 \leq \cos(\eta) \leq 1, \text{ where } \|\zeta(\eta)\| = 29.3827994, \quad \|\Psi_\zeta(\eta)\|_c = 0.05 < \delta, \ \forall\, \eta \geq 0.$$

Moreover, the singular values of $\mathscr{A}$ *and* $A_1 + A_2$ *are*

$$\sigma_{\max}(\mathscr{A}) = 6.24294338387, \qquad \sigma_{\max}(A_1 + A_2) = 1.41421356237.$$

Applying Theorem 5.1 on (74), we get

$$\eta = 0.02702993872.$$

Finally, time-varying delays $d_1(\eta)$ *and* $d_2(\eta)$ *for* $\eta = 0.02702993872$ *are,*

$$d_1(\eta) = 0.9999999635 \text{ and } d_2(\eta) = 0.9999999973,$$

where η *is the estimated time of FTS.*

Example 8.2. *In this example, we will discuss the finite-time stability of the system,*

$$\begin{cases} {}^C D^\alpha_\eta \zeta(\eta) & = \mathscr{A}\zeta(\eta) + \mathscr{B}\mathbf{u}(\eta) + E\mathbf{w}(\eta) + \sum_{k=1}^{m} A_k \zeta(\eta - d_k(\eta)) + \kappa, \\ y(\eta) & = \mathscr{C}\zeta(\eta), \qquad \eta \in [0, 1], \end{cases} \tag{77}$$

using Theorem 5.1, with the following assumptions:

$$\mathscr{A} = \begin{bmatrix} 4 & -2 \\ 3 & 5 \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} 5.9 \\ -1 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} 4 & 2 \end{bmatrix}, \quad \zeta_0 = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix},$$
$$E = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \kappa = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2\pi} \\ 12\sqrt{\pi} \end{bmatrix},$$

where $Y_1 = 2, \sigma_1 = 2, a_1 = 2$ *and* $Y_2 = 4, \sigma_2 = 3, a_2 = 1$*. Firstly, we have obtained the solution of system (77) for the considered values, and further, we will discuss the finite-time stability for this system as well.*

For $p = 0$ *and* $\alpha = 0.3$*, we have from Definition 3.2,*

$$\varphi(1 - \xi) = \frac{I(1 - \xi)^{-0.7}}{\Gamma(0.3)}, \qquad \varphi_0(\eta) = I_{2\times 2}.$$

Also, by substituting values,

$$\zeta(\xi - d_1(\xi)) = \begin{bmatrix} 0.02\xi - \cos(0.02\xi) \\ 0.02\xi - \cos(0.02\xi) \end{bmatrix}, \qquad \zeta(\xi - d_2(\xi)) = \begin{bmatrix} 0.01\xi - 2\sin(0.01\xi) \\ 0.01\xi - 2\sin(0.01\xi) \end{bmatrix},$$

$$\mathbf{u}(\boldsymbol{\eta}) = -\eta + 1.9, \qquad w(\eta) = \begin{bmatrix} -4 \\ -\eta - 6 \end{bmatrix},$$

in (77), we obtain the approximate solution,

$$\begin{bmatrix} \zeta_1(\eta) \\ \zeta_2(\eta) \end{bmatrix} = \begin{bmatrix} (6.0032\eta - 24.12 - 3.343\cos(0.02\eta) - 4.5\sin(0.01\eta)) \\ \times(\eta-1)^{\frac{3}{10}} + (25.9 + 3.343\cos(0.02\eta) \\ +4.5\sin(0.01\eta))\eta^{\frac{3}{10}} - 6.0032\eta^{\frac{13}{10}} \\ (-1.7\eta - 28.8 - 2.23\cos(0.02\eta) + 4.5\sin(0.01\eta)) \\ \times(\eta-1)^{\frac{3}{10}} + (28.3 + 2.23\cos(0.02\eta) \\ -4.5\sin(0.01\eta))\eta^{\frac{3}{10}} - 0.5 + 1.7\eta^{\frac{13}{10}} \end{bmatrix}. \tag{78}$$

The graph of the solution of (77) is illustrated in Figure 6,

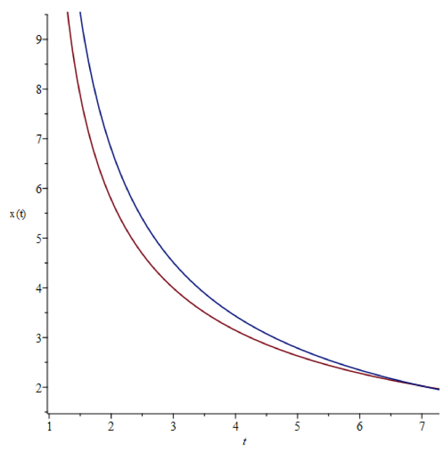

Figure 6: Graphical analysis of the solution of the fractional-order model in (77).

We have obtained Figure 7(a) with the fracture term $\kappa = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2\pi} \\ 6\sqrt{\pi} \end{bmatrix}$*, and Figure 7(b) with fracture term* $\kappa = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 8\sqrt{2\pi} \\ 6\sqrt{\pi} \end{bmatrix}$*, which are given below,*

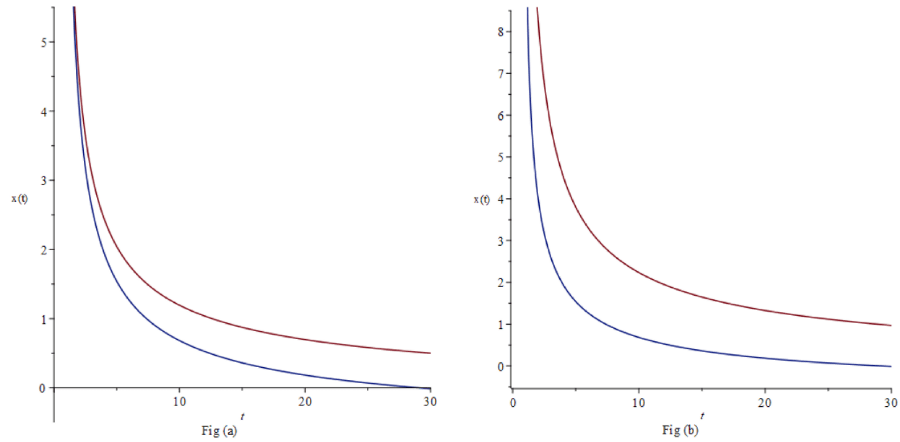

Figure 7: Graphical analysis of the fractional-order model in (77) with different values of fracture.

The comparison graphs of the stability of the fractional/ non-integer order model approaching classical order are shown in Figure 8 as follows:

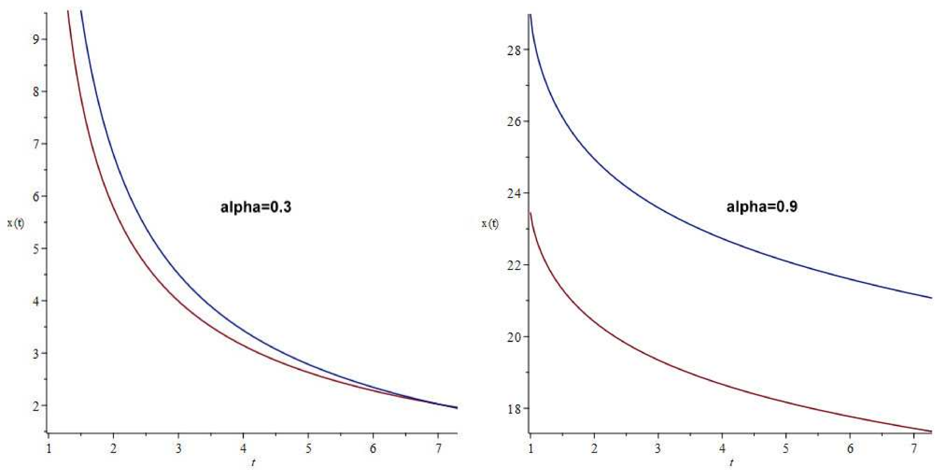

Figure 8: Graphical analysis of the fractional-order model in (77) with different values of α.

In (77), if the fractional-order derivative is replaced by the classical order derivative, we get

$$\begin{cases} \dfrac{d}{d\eta}\zeta(\eta) & = \mathscr{A}\zeta(\eta) + \mathscr{B}\mathbf{u}(\eta) + E\mathbf{w}(\eta) + \displaystyle\sum_{k=1}^{m} A_k\zeta(\eta - d_k(\eta)) + \kappa, \\ y(\eta) & = \mathscr{C}\zeta(\eta), \qquad \eta \in [0,1]. \end{cases} \tag{79}$$

Then, the graph of the solution of (79) is illustrated in Figure 9,

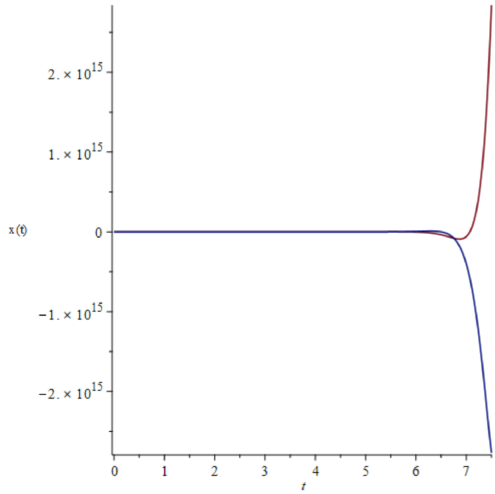

Figure 9: Graphical analysis of the solution of the classical-order model in (79).

It is seen that when system (77) is incorporated with a classical derivative, then we get a clearly unstable solution graph.

The presence of external disturbances or uncertainties affects the robustness of the system's stability. In Figure 10(a), we have $w(\eta) = \begin{bmatrix} 4 \\ \eta + 6 \end{bmatrix}$ *and in Figure 10(b), we have* $w(\eta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. *From Figures 10(a) and 10(b), it can be seen that the robustness of the system stability is affected.*

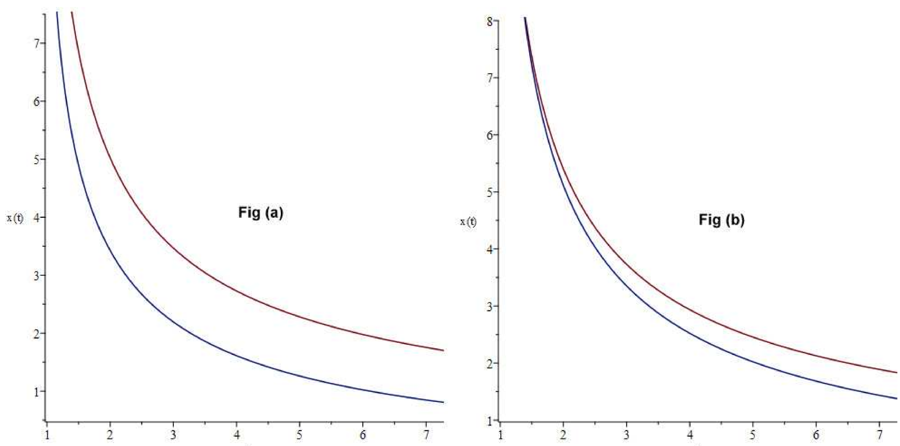

Figure 10: Graphical analysis of the comparison of the external disturbances of the fractional-order model in (77).

Now, we will check the FTS using Theorem 5.1 with respect to

$$\eta_0 = 0, \quad J = \{0, 1\}, \quad \delta = 0.6, \quad \varepsilon = 7, \quad a_u = 0.1, \quad a_w = 8, \quad p = 0, \textit{ for } -1 \leq \cos(\eta) \leq 1,$$
$$-1 \leq \sin(\eta) \leq 1, \quad \textit{where } \|\zeta(\eta)\| = 6.79590821, \quad \|\Psi_\zeta(\eta)\|_c = 0.5 < \delta, \ \forall\, \eta \geq 0.$$

Moreover, the singular values of $\mathscr{A}$ *and* $A_1 + A_2$ *are*

$$\sigma_{\max}(\mathscr{A}) = 5.85492185168, \qquad \sigma_{\max}(A_1 + A_2) = 4.12310562562.$$

Applying the condition of Theorem 5.1 on (77), we get

$$\eta = 0.1195804554.$$

Finally, time-varying delays $d_1(\eta)$ and $d_2(\eta)$ for $\eta = 0.1195804554$ are

$$d_1(\eta) = 0.9999971401 \text{ and } d_2(\eta) = 0.002391608538,$$

where η is the estimated time of FTS.

Example 8.3. *In this example, we will discuss the controllability and observability of the system,*

$$\begin{cases} {}^C D_\eta^\alpha \zeta(\eta) & = \mathscr{A}\zeta(\eta) + \mathscr{B}\mathbf{u}(\eta) + E\mathbf{w}(\eta) + \sum_{k=1}^{m} A_k \zeta(\eta - d_k(\eta)) + \kappa, \\ y(\eta) & = \mathscr{C}\zeta(\eta), \qquad \eta \in [0,1]. \end{cases} \tag{80}$$

Using Theorems 6.1 and 7.1, with the following assumptions:

$$\mathscr{A} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} 2 & 4 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} -2 & -4 \\ 5 & -1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ -0.5 & 1 \end{bmatrix}, \quad \kappa = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 14\sqrt{2\pi} \\ 6\sqrt{2\pi} \end{bmatrix},$$

where $Y_1 = 7, \sigma_1 = 2, a_1 = 2$ and $Y_2 = 2, \sigma_2 = 3, a_2 = 2$ as defined in (2).

By using the Gramian matrix, we utilize our criteria to investigate the controllability and observability of system (80).

For $\alpha = 1.5$ and $p = 0$, Definition 3.2 gives

$$\varphi(1-\xi) = \begin{bmatrix} \frac{(1-\xi)^{0.5}}{\Gamma(1.5)} & 0 \\ 0 & \frac{(1-\xi)^{0.5}}{\Gamma(1.5)} \end{bmatrix}.$$

Substituting all the values in the controllability Gramian matrix given in Theorem 6.1 and for $\eta_f = 1$, we have

$$W_c[0,1] = \begin{bmatrix} 10.18591636 & 5.092958178 \\ 5.092958178 & 2.546479089 \end{bmatrix}.$$

One can verify that $W_c[0,1]$ is invertible, symmetric, positive semi-definite, and positive definite. So, the above system (80) is controllable.

For $\alpha = 1.5$ and $p = 0$, Definition 3.2 gives

$$\varphi_0(\eta) = I_{2\times 2}.$$

Substituting all the values in the observability Gramian matrix given in Theorem 7.1 and for $\eta_f = 1$, we have

$$W_o[0,1] = \begin{bmatrix} \frac{14}{117} & -\frac{17}{117} \\ -\frac{17}{117} & \frac{29}{117} \end{bmatrix}.$$

One can verify that $W_o[0,1]$ is invertible, symmetric, positive semi-definite, and positive definite. So, the above system (80) is observable.

Example 8.4. *In this example, we will discuss the controllability and observability of the system,*

$$\begin{cases} {}^{C}D_{\eta}^{\alpha}\zeta(\eta) & = \mathscr{A}\zeta(\eta) + \mathscr{B}\mathbf{u}(\eta) + E\mathbf{w}(\eta) + \sum_{k=1}^{m} A_k\zeta(\eta - d_k(\eta)) + \kappa, \\ y(\eta) & = \mathscr{C}\zeta(\eta), \qquad \eta \in [0,1], \end{cases} \tag{81}$$

using Theorems 6.1 and 7.1, with the following assumptions:

$$\mathscr{A} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} 4 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} -2 & -4 \\ 5 & -1 \end{bmatrix},$$
$$A_1 = \begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ -0.5 & 1 \end{bmatrix}, \quad \kappa = \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = \begin{bmatrix} 14\sqrt{2\pi} \\ 6\sqrt{2\pi} \end{bmatrix},$$

where $Y_1 = 7, \sigma_1 = 2, a_1 = 2$ and $Y_2 = 2, \sigma_2 = 3, a_2 = 2$ as defined in (2).

By using the Gramian matrix, we utilize our criteria to investigate the controllability and observability of system (81).

For $\alpha = 1.5$ and $p = 0$ to $p = 1$, Definition 3.2 gives

$$\varphi(1-\xi) = \begin{bmatrix} \frac{(1-\xi)^{0.5}}{\Gamma(1.5)} + \frac{(1-\xi)^{1.5}}{\Gamma(3)} & \frac{-2(1-\xi)^{1.5}}{\Gamma(3)} \\ \frac{3(1-\xi)^{1.5}}{\Gamma(3)} & \frac{(1-\xi)^{0.5}}{\Gamma(1.5)} + \frac{-4(1-\xi)^{1.5}}{\Gamma(3)} \end{bmatrix}.$$

Substituting all the values in the controllability Gramian matrix given in Theorem 6.1 and for $\eta_f = 1$, we have

$$W_c[0,1] = \begin{bmatrix} 10.18591636 & 8.101969290 \\ 8.101969290 & 6.555490201 \end{bmatrix}.$$

One can verify that $W_c[0,1]$ is invertible, symmetric, positive semi-definite, and positive definite. So, the above system (81) is controllable.

For $\alpha = 1.5$ and $p = 0$ to $p = 1$, Definition 3.2 gives

$$\varphi_0(\eta) = \begin{bmatrix} 1 + \eta^{1.5} & -2\eta^{1.5} \\ 3\eta^{1.5} & 1 - 4\eta^{1.5} \end{bmatrix} \frac{1}{\Gamma(2.5)}.$$

Substituting all the values in the observability Gramian matrix given in Theorem 7.1 and for $\eta_f = 1$, we have

$$W_o[0,1] = \begin{bmatrix} 54.21919496 & -27.87501859 \\ -27.87501859 & 20.95892039 \end{bmatrix}.$$

One can verify that $W_o[0,1]$ is invertible, symmetric, positive definite, and positive semi-definite. So, the above system (81) is observable.

Example 8.5. *In this example, we will discuss the controllability and observability of the system,*

$$\begin{cases} {}^{C}D_{\eta}^{\alpha}\zeta(\eta) & = \mathscr{A}\zeta(\eta) + \mathscr{B}\mathbf{u}(\eta) + E\mathbf{w}(\eta) + \sum_{k=1}^{m} A_k\zeta(\eta - d_k(\eta)) + \kappa, \\ y(\eta) & = \mathscr{C}\zeta(\eta), \qquad \eta \in [0,1], \end{cases} \tag{82}$$

using Theorems 6.2 and 7.2, with the following assumptions:

$$\mathscr{A} = \begin{bmatrix} 0.4 & 0.5 \\ 0.22 & 3 \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} 2 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} -2 & -1 \\ 2 & 3 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 \\ -2 & -0.5 \end{bmatrix}, \quad \kappa = \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = \begin{bmatrix} 14\sqrt{2\pi} \\ 6\sqrt{2\pi} \end{bmatrix},$$

where $Y_1 = 7, \sigma_1 = 2, a_1 = 2$ *and* $Y_2 = 2, \sigma_2 = 3, a_2 = 2$ *as defined (2).*

By using the Kalman test, we now utilize our criteria to investigate the controllability and observability of system (82).

For controllability, we have from Theorem 6.2,

$$rank\left[\mathscr{B}|\mathscr{A}\mathscr{B}|E|\mathscr{A}E|\sum_{k=1}^{2} A_k|\mathscr{A}\sum_{k=1}^{2} A_k|\kappa|\mathscr{A}\kappa\right] = 2 = n,$$

which is required. So, system (82) is controllable.

For observability, we have from Theorem 7.2,

$$rank\begin{bmatrix} \mathscr{C} \\ \mathscr{C}\mathscr{A} \end{bmatrix} = rank\begin{bmatrix} 2 & 1 \\ 1.02 & 4 \end{bmatrix} = 2 = n.$$

Hence, the system (82) is observable.

9 Discussion

The stability of dynamical systems with time-varying delays is an important research topic because their occurrence cause instability and oscillation. Various real-world applications of the dynamical model can be found in neural networks [21], power systems [47], robotic systems [56], chronic myelogenous leukemia [64], electric vehicles [67], and servo systems [72].

We have undertaken numerical simulations to compare our work with existing works [29, 47]. The problem of the power system with multiple time delays is investigated in [47]. With the considered values in Example 8.1, the stability graph of the classical order model in [47] is shown in Figure 11(a) and the fractional-order model (74) is shown in Figure 11(b) without the fracture term.

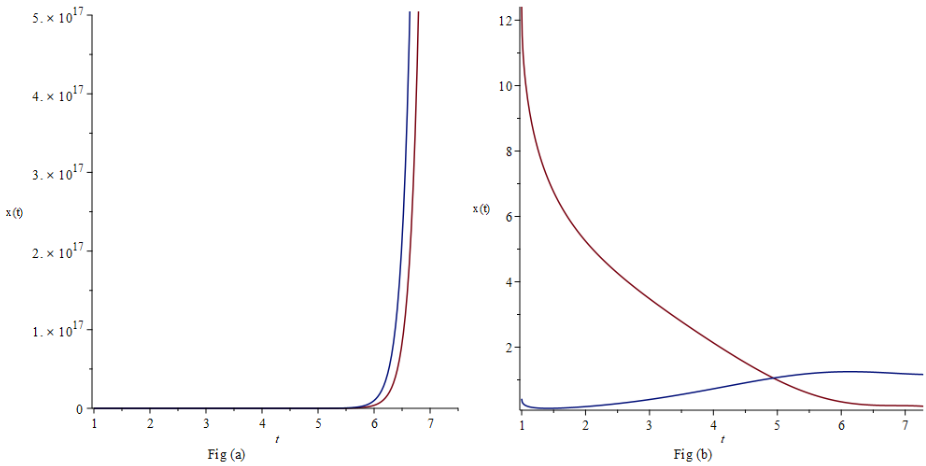

Figure 11: Comparison of the stability of the classical order model in [47] and the fractional-order model (74) without fracture term.

With the considered values in Example 8.1, the stability graphs of the classical order model in [29] are shown in Figure 12(a), and the fractional-order model (74) is shown in Figure 12(b) with a single delay and without a fracture term.

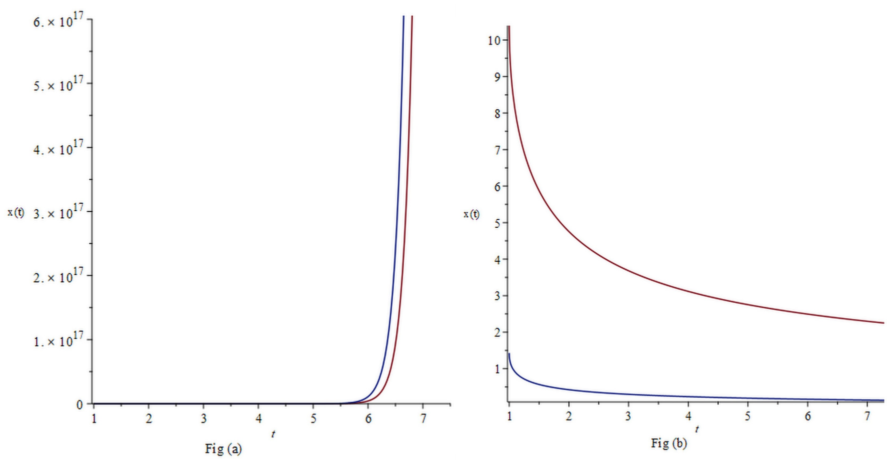

Figure 12: Comparison of the stability of the classical order model in [29] and the fractional-order model (74) with single delay and without fracture term.

The numerical simulations have been used to benchmark the algorithm for comparison of our model in (1) with different models in [29, 47]. The comparison results obtained by necessary numerical simulations highlight the superiority of the fractional-order model (1) in terms of stability, whereas the classical order model [29] and [47] exhibits instability, underscoring the effectiveness of our proposed approach in real-world scenarios.

The simulations have been given by numerical examples, specifically illustrating stability and performance, which can be seen in Figures 1–3 and Figures 6–8, which validates our work. In Example 8.1, the stable solution of system (74) can be seen in Figure 1. Graphical analysis of the fractional-order model in (74) with different values of α can be seen in Figure 2. The comparison graphs of the stability of the fractional/non-integer order model approaching classical order are shown in Figure 3. It can be observed that when system (74) is incorporated with a classical derivative, then we get a clearly unstable solution graph, and the graph of the solution of (76) is given in Figure 4. The presence of external disturbances or uncertainties affects the robustness of the system's stability, which can be seen in Figure 5. In Figure 5(a), we have $w(\eta) = \begin{bmatrix} 1.4 \\ \eta \end{bmatrix}$ and in Figure 5(b), we have $w(\eta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. From Figures 5(a) and 5(b), it can be seen that the robustness of the system stability is affected.

In Example 8.2, the stable solution of system (77) can be seen in Figure 6. Graphical analysis of the fractional-order model in (77) with different values of fracture can be seen in Figure 7. The comparison graphs of the stability of the fractional/non-integer order model approaching classical order are shown in Figure 8. It can be observed that when system (77) is incorporated with a classical derivative, then we get a clearly unstable solution graph, and the graph of the solution of (79) is given in Figure 9. The presence of external disturbances or uncertainties affects the robustness of the system's stability, which can be seen in Figure 10. In Figure 10(a), we have $w(\eta) = \begin{bmatrix} 4 \\ \eta + 6 \end{bmatrix}$ and in Figure 10(b), we have $w(\eta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. From Figures 10(a) and 10(b), it can be seen that the robustness of the system stability is affected.

The controllability and observability of the systems (80), (81), and (82) is numerically explained in Examples 8.3, 8.4, and 8.5, respectively. In Examples 8.3 and 8.4, we have incorporated the Gramian matrix to explore the controllability and observability of the system (80) and (81), respectively. In Example 8.5, we have utilized the Kalman test to investigate the controllability and observability of the system (82).

10 Conclusion

This study explores the stability and control properties of fractional-order dynamical systems with time-varying delays and fracture. It also highlights the critical role of the fractional-order parameter α in influencing the system behavior. Through theoretical analysis and numerical simulations, we have demonstrated that the fractional-order models achieve stability more effectively than the integer-order models, which is shown by comparative results in Examples 8.1 and 8.2. Furthermore, the proposed criteria for finite-time stability and control provide practical insights that are applicable to diverse engineering domains, including civil engineering, robotics, and aerospace. The numerical validations confirm the efficiency of our approach, offering researchers and engineers a robust methodology to achieve desired system performance within finite time intervals. By bridging theoretical advancements with real-world applications, this work contributes significantly to the understanding and optimization of fractional-order dynamical systems, paving the way for future innovations in control engineering and applied mathematics.

Acknowledgement I acknowledge the continuing support of my research by The Women University Multan, Pakistan.

Conflicts of Interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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